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DYNAMIC PROGRAMMING, NONLINEAR VARIATIONAL PROCESSES, AND SUCCESSIVE APPROXIMATIONS

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1. INTRODUCTION

Our object in this paper is to show that a blend of dynamic programming, successive approximations and digital computers enables us to approach various classes of nonlinear variational problems formerly far beyond our reach.

The general problem we wish to consider is one which arises in many parts of analysis, mathematical physics and applied mathematics, namely that of determining the minimum of a functional of the form

$$J(v) = \int_0^T F(x_1(t),\ldots,x_N(t))dt + G(x_1(T),\ldots,x_N(T)), \qquad (1.1)$$

over all forcing functions $v_i(t)$ subject to relations of the type

a. $\dfrac{dx_i}{dt} = H_i(x_1,x_2,\ldots,x_N) + v_i(t), \quad x_i(0) = c_i, \quad i = 1,2,\ldots,N$

b. $\int_0^T K_j(v_1,v_2,\ldots,v_N)dt \leq b_j, \quad j = 1,2,\ldots,L \qquad (1.2)$

c. $p_i(x_1,\ldots,x_N;t) \leq v_i(t) \leq q_i(x_1,\ldots,x_N;t), \quad 0 \leq t \leq T.$

As we have shown elsewhere [1,2], the computational solution of questions of this type can be transformed by means of the functional equation technique of dynamic programming into problems

involving the determination of sequences of functions of N variables using elementary operations. If, however, $N \geq 3$, the limited memory and speed of current digital computers prevents this from being a routine technique of solution at the present time.

It follows that if we wish to attack general problems of this nature, we must introduce more refined techniques.

2. THE LINEAR CASE

The starting point of our investigation is the result contained in an earlier paper on control processes [3] in which it is shown that in the case where the underlying equations of (1.2a) are linear, the criterion functions F and G are linear, and the constraints of (1.2c) have the simpler form

$$p_1(t) \leq v_1(t) \leq q_1(t), \quad 0 \leq t \leq T, \tag{2.1}$$

then the variational problem described in the preceding section can be reduced to the computation of a sequence of functions of one variable, regardless of the size of N. If the quantity L appearing in (1.2b) is equal to one, we can do this directly; if it is greater than one, we employ Lagrange multipliers [4].

It was also shown that if the functions F and H are linear, and G is a function of only k of the N components of $x(T)$, then a computational solution can be effected in terms of functions of k variables.

This fact is important in connection with engineering control

processes involving time-lags and economic control processes involving a complex of industries.

These results combined with the classical tool of successive approximations will permit us to treat various parts of the general problem.

3. SUCCESSIVE APPROXIMATIONS

Let us now outline how the results obtained for the linear case, aided and abetted by successive approximations, provide a feasible scheme for computing the solutions of nonlinear problems, in terms of sequences of functions of one variable.

We shall consider first the case where we are interested in terminal control with G linear in its arguments. Let $v_i^0(t)$, $i = 1, 2, \ldots, N$, be an initial choice of forcing function satisfying (1.2a) and (1.2b) and let the functions $x_i^0(t)$ be determined by means of the equations

$$\frac{dx_i}{dt} = H_i(x_1, \ldots, x_N) + v_i^0(t), \quad x_i(0) = c_i, \qquad i = 1, 2, \ldots, N. \tag{3.1}$$

To obtain the next approximation, $v_i^1(t)$, consider the problem of minimizing $G(x_1(T), x_2(T), \ldots, x_N(T))$ over all functions $v_i^1(t)$ satisfying (1.2a) and (1.2b) where the functions $x_i(t)$ are now determined by the approximating linear relations

$$\frac{dx_i}{dt} = H_i(x_1^0,\ldots,x_N^0) + \sum_{j=1}^{N}(x_j - x_j^0)\frac{\partial H_i}{\partial x_j^0} + v_i(t),$$
$$x_i(0) = c_i. \tag{3.2}$$

Since these equations are linear, and G has been assumed to be linear, the minimizing sequence, v_i^1, can now be computed via sequences of functions of one variable. In some cases, the solution can be obtained analytically, cf. [5,6], thus greatly simplifying the application of this method.

Once the new forcing functions, $v_i^1(t)$, have been determined, the new state functions, $x_i^1(t)$, are determined by way of (3.1), with v_i^0 replaced by v_i^1.

Repeating this process, step-by-step, we determine a sequence of forcing functions, $\left\{v_i^k(t)\right\}$, and a sequence of state functions, $\left\{x_i^k(t)\right\}$.

4. MONOTONICITY OF APPROXIMATION

Without prejudicing ourselves in the matter of convergence, it is easy to show that we have monotonocity of approximation in the sense that

$$G(x_1^0(T),\ldots,x_N^0(T)) \geq G(x_1^1(T),\ldots,x_N^1(T)) \geq \cdots$$
$$\geq G(x_1^k(T),\ldots,x_N^k(T)) \geq \cdots \tag{4.1}$$

To see this, observe that if $v_i(t)$ is taken equal to $v_i^0(t)$ in (3.2), we obtain a system of linear equations whose solution

is clearly $x_1 = x_1^0$. It follows that a set of forcing functions v_1 which minimize must yield a value of $G(x_1(T),\ldots,x_N(T))$ which is at most $G(x_1^0(T),\ldots,x_N^0(T))$. The general result follows inductively.

This monotonicity of approximation is not surprising, considering that we are essentially carrying out an approximation in policy space[7].

5. NONLINEAR CRITERION FUNCTION—I

To begin with, we note that the general problem described in (1.1) can always be reduced to a terminal control process by the introduction of a new dependent variable x_{N+1} defined by the differential relation

$$\frac{dx_{N+1}}{dt} = F(x_1,x_2,\ldots,x_N), \quad x_{N+1}(0) = 0. \tag{5.1}$$

If G is nonlinear, but contains only k of the N components of $x(T)$, then, as indicated above, the approximation procedure reduces the computational problem to one involving sequences of functions of k variables.

6. NONLINEAR CRITERION FUNCTION—II

Alternatively, we can approximate to G not only by means of linear terms, but more accurately by means of a second degree expression. Having done this, we can take advantage of the fact that the minimization of a quadratic functional of the form

$$J(v) = \int_0^T [(x,B(t)x) + (x,b(t)) + (v,C(t)v) + (v,c(t))]\,dt + (x(T),Gx(T)) + (x(T),g) \tag{6.1}$$

where x and v are connected by means of a linear equation

$$\frac{dx}{dt} = A(t)x + v, \quad x(0) = c, \tag{6.2}$$

with no other constraints, depends upon the solution of linear equations; cf. [5] where classical techniques are employed. Dynamic programming techniques, utilizing the fact that the minimum is a quadratic function of the c_i, the components of c, can also be applied.

This affords a new approach to some general classes of nonlinear equations and nonlinear variational problems which we will discuss in greater detail subsequently.

7. DISCUSSION

There are a number of other types of approximations which we cannot present here due to limitations of space. There are also a large number of interesting analytic questions concerning convergence, local minima, rapidity of convergence, etc., to be discussed at a later date in connection with the foregoing techniques.

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